<u>Exam</u> Will be oral

Admission: "Intermediate Exam"

i.e. Exercise sheet to hand in

after ~ 2/3 of course

<u>Related Courses</u>

- .) AG II (Huybrechts)
- .) Finite grp schemes (Martin)
- .) Étale cohomology (Ivanov)
- .) Automorphic Forms (Jana)

<u>Last time</u> k field

1) EC over k $\underset{\text{def}}{=}$ proper smooth connected one-dim grp sch $E = (E, +)/\operatorname{Spec} k$

2) ECs are commutative

3) $\{$Prop sm curves $/\operatorname{Spec} \mathbb{C}\} \simeq \{$compact R.S.$\}$

4) All ECs$/\mathbb{C}$ of form $\mathbb{C}/\Lambda$, $\Lambda \subseteq \mathbb{C}$ lattice

<u>Today</u> Expand on 1) & 3)

§1 Kähler Differentials (= differential 1-forms)

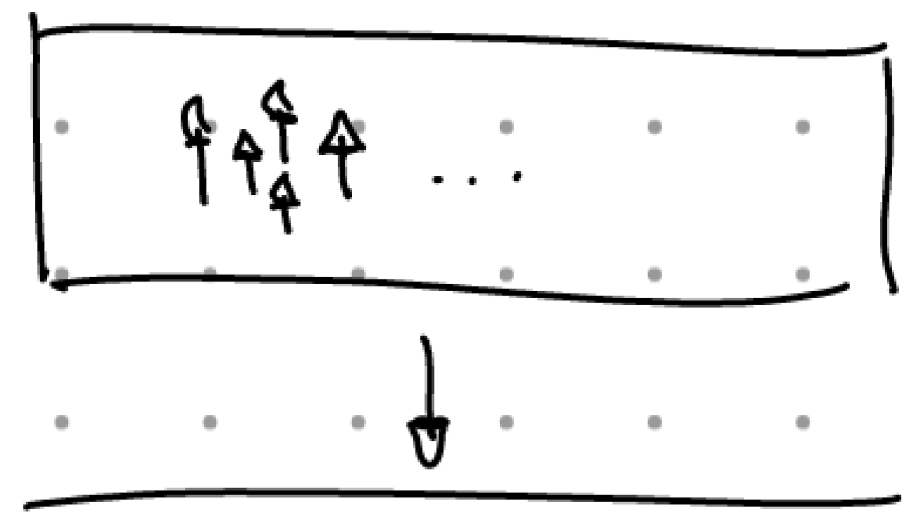
Def R rg., A R-alg., M A-module

1) R-derivation from A to M $\underset{\text{def}}{=}$ (Leibniz)

R-linear $\delta : A \to M$ s.th. $\delta(fg) = f\,\delta g + g\,\delta f$

Intuition
$M = A$

$$\begin{array}{c} \operatorname{Spec} A \\ \downarrow \\ \operatorname{Spec} R \end{array}$$



vector field w/ directions in fibres.

E.g. $R = k[X]$, $A = k[X,Y]$, $M = A$. Let $g \in A$, set

$$\delta f := g(X,Y) \cdot \frac{\partial f}{\partial Y}(X,Y).$$

A-module

2) Universal R-derivation $\underset{\text{def}}{=}$ pair $(\Omega^1_{A/R}, d)$

$d : A \to \Omega^1_{A/R}$ R-derivation

s.th. $\forall$ R-derivat. δ

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[dr, "\delta"'] & \Omega^1_{A/R} \arrow[d, dotted, "\varphi"] \\ & M \end{tikzcd}$$

$\exists!$ A-linear φ s.th. $\delta = \varphi \circ d$.

Lemma $(\Omega^1_{A/R}, d)$ exists, is unique up to unique iso.

Prf

Set $$\Omega^1_{A/R} := \frac{\bigoplus_{a \in A} A \cdot da}{\langle d(ra) = r\,da,\ d(fg) = f\,dg + g\,df,\ d(f+g) = df + dg \rangle}$$

define $d : A \to \Omega^1_{A/R}$, $a \mapsto da$.

See yourself: d is R-derivation + universal. $\square$

E.g. Let $A = R[T_1, \dots, T_n]$.

Then $\left(\Omega^1_{A/R} = \bigoplus_{i=1}^{n} A \cdot dT_i \,,\; df := \frac{\partial f}{\partial T_i} \cdot dT_i \right)$

is universal. (Try yourself!) Free of rk n!

Def $(\Omega^1_{A/R}, d)$ module of Kähler differentials

Intuition Vector fields along fibres in $\operatorname{Spec} A \to \operatorname{Spec} R$

$\rightsquigarrow \operatorname{Der}_R(A, A) = \operatorname{Hom}_A(\Omega^1_{A/R}, A)$

so $\Omega^1_{A/R}$ is dual of Tangent bundle.

Universality implies functoriality: $\varphi: A \to B$ map of R-algs.

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "d_A"'] & B \arrow[d, "d_B"] \\ \Omega^1_{A/R} \arrow[r, dashed] & \Omega^1_{B/R} \end{tikzcd}$$

$\exists!$ A-linear

$\Rightarrow$ $B \otimes_A \Omega^1_{A/R} \longrightarrow \Omega^1_{B/R}$ B-linear

$1 \otimes da \longmapsto d\varphi(a)$. @

Pullback of differentials for $\operatorname{Spec} B \to \operatorname{Spec} A$.

Special case: $A \twoheadrightarrow A/I$. Then $\Omega^1_{A/R} \twoheadrightarrow \Omega^1_{A/I\,|\,R}$

get ex seq $I/I^2 \longrightarrow A/I \otimes_A \Omega^1_{A/R} \longrightarrow \Omega^1_{A/I\,/R} \longrightarrow 0$.

$f \longmapsto 1 \otimes df$.

E.g. 1) $A = R[T_1, \dots, T_n] / (f_1, \dots, f_m)$

Then $\Omega^1_{A/R} = \bigoplus_{i=1}^n A\, dT_i \Big/ (df_j)_{j=1}^m$

But $df_j = \sum_{i=1}^n \frac{\partial f_j}{\partial T_i} \cdot dT_i$, so get presentation

$$A^m \xrightarrow{J(f)} A^n \longrightarrow \Omega^1_{A/R} \longrightarrow 0$$

with $J(f) = \left(\frac{\partial f_j}{\partial T_i} \right)_{i,j}$ Jacobian matrix.

2) $R = \mathbb{Z}$, $A = \mathbb{Z}[T]/f(T)$ order in number field, say.

Then $\Omega^1_{A/R} = A\,dT \Big/ f'(T) \cdot dT = A/(f'(T)) \cdot dT$

i.e. $\cong A/\text{Different}(A/R)$

3) $k = \mathbb{F}_p(t)$, $K = k[T]/T^p - t$ (insep. field ext)

$\Omega^1_{K/k} \simeq K\,dT \Big/ (T^p - t)'(T) = K\,dT$

$\neq 0$, free rk 1 / K

Note: $K \otimes_k K \simeq K[\varepsilon]/(\varepsilon^p)$ non-reduced!

Try yourself: K/k alg is separable $\iff \Omega^1_{K/k} = 0$.

Freely used the following: $\exists$ canonical isos

·) $S^{-1}\Omega^1_{A/R} \xrightarrow{\sim} \Omega^1_{S^{-1}A/R}$

·) $R' \otimes_R \Omega^1_{A/R} \xrightarrow{\sim} \Omega^1_{R'\otimes_R A/R'}$

Consequence:

·) For $X \to S$ morph of sch, local Ω's glue to q-coh $\mathcal{O}_X$-module $\Omega^1_{X/S}$

·) For $f: X \to Y$ over S, get $f^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

(cf. @ on p. 3)

·) For $S' \to S$, $f: X' = S' \times_S X \to X$, get

$$f^*\Omega^1_{X/S} \xrightarrow{\sim} \Omega^1_{X'/S'}.$$

§2 Smoothness Idea For $X/\operatorname{Spec} k$ loc. of f.t.,

X smooth in x $\leadsto$ "looks like $\mathbb{A}^d$ near x" $(d = \dim_x X)$

$\leadsto$ $\exists\, z_1, \dots, z_d \in \mathcal{O}_{X,x}$ s.th.

$$\left(z^* \Omega^1_{\mathbb{A}^d/k}\right)_x \xrightarrow{\sim} \Omega^1_{X/k,x}$$

For manifolds, would mean that Jacobian of $(z_1, \dots, z_d)$ invertible near x, i.e. that $(z_1, \dots, z_d)$ define local chart.

<u>Clear</u> z exists $\Leftrightarrow$ $\Omega^1_{X/k,x}$ free rank d

<u>Def</u> $X \xrightarrow{f} \operatorname{Spec} k$ smooth $\underset{def}{\Leftrightarrow}$ f is loc. of f.t. and

$\forall x \in X$, $\Omega^1_{X/k,x}$ free of rk $= \dim_x X$ over $\mathcal{O}_{X,x}$

<u>Clear</u> local on X.

<u>E.g.</u> $k = \mathbb{F}_p(t)$, $K = k[t^{1/p}]$ $\operatorname{Spec} K \to \operatorname{Spec} k$ <u>not</u> smooth.

<u>Try yourself</u>: X 0-dim, loc. of f.t. $/k$ is smooth

$\Leftrightarrow$ $X = \coprod \operatorname{Spec} K_i$ K_i/k finite separable.

Prop Let $U = V(I) \overset{i}{\subseteq} \mathbb{A}^n_k$ closed. Equivalent:

1) $U \to \operatorname{Spec} k$ smooth
2) $U_{\bar{k}}$ is regular
3) The following sequence is exact & locally split

$$0 \to I/I^2 \to i^* \Omega^1_{\mathbb{A}^n} \to \Omega^1_{U/k} \to 0 \qquad @$$

4) For any choice of generators $(g_1, \dots, g_r) = I$, $\forall x \in U$,

$$\left(\frac{\partial g_j}{\partial T_k} \right)_{\substack{k=1,\dots,n \\ j=1,\dots,r}} (x) \in M_{n\times r}(\kappa(x))$$

has rank $n - \dim_x U$.

<u>Rmks</u> ·) Noeth loc ring (R, m) regular $\underset{def}{=}$

Krull dim $R = \dim_{R/m} m/m^2$.

·) loc noeth X regular $\underset{def}{=}$ $\mathcal{O}_{X,x}$ regular $\forall x$.

·) 2) called <u>geometrically regular</u>

4) called <u>Jacobian criterion</u>

·) 1) & 2) are intrinsic, hence 3) & 4) indep of presentation!

<u>Proof</u> General observation (Try yourself!)

All statements may be shown after $\bar{k} \otimes_k -$.

$\longrightarrow$ wlog $k = \bar{k}$ in following.

Useful because $\forall\, x \in U(k)$ <u>rational</u> points

<u>Try yourself</u>: $\mathrm{Der}_k(\mathcal{O}_{U,x}, k) = \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$.

(when $\kappa(x) = k$.)

| Equivalently: $\mathfrak{m}_x/\mathfrak{m}_x^2 \xrightarrow{\sim} \kappa(x) \otimes_{\mathcal{O}_{U,x}} \Omega^1_{U,x}$.

(Compare: $x = \mathrm{Spec}\, \mathbb{F}_p(t)[T]/_{T^p - t} \longrightarrow \mathbb{A}^1_{\mathbb{F}_p(t)}$. Then

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \ni T^p - t \longmapsto 0 \in \Omega^1_{\kappa(x)/k}\ .)$$

In following $d_x := \dim_x U$.

Proof $1) \Rightarrow 2)$ Regularity stable under localization

$\Rightarrow$ Enough to show for $\mathcal{O}_{U,x}$, $x \in U(k)$ closed.

But $\dim_k \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim_k \kappa(x) \otimes_{\mathcal{O}_{U,x}} \Omega^1_{U,x} = d_x$

by assumption on Ω^1. $\square$

$2) \Rightarrow 3)$ Enough to show exactness

$$0 \to I_x/I_x^2 \to i^* \Omega^1_{\mathbb{A}^n, x} \to \Omega^1_{U,x} \to 0 \quad \forall x \in U(k).$$
$$@$$

Given x, pick $g_{d+1}, \dots, g_n \in I_x$ generating

$$\ker\left(\mathfrak{n}_x/\mathfrak{n}_x^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2\right)$$

(Here $\mathfrak{n}_x \subseteq \mathcal{O}_{\mathbb{A}^n,x}$, $\mathfrak{m}_x \subseteq \mathcal{O}_{U,x}$)

By Nakayama, generate I_x/I_x^2

$\Rightarrow \dim_k \left(\kappa(x) \otimes_{\mathcal{O}_{U,x}} I_x/I_x^2\right) \leq n-d.$

$\Rightarrow$ Seq. @ exact after $\kappa(x) \otimes_{\mathcal{O}_{U,x}} -$ for $\dim_{\kappa(x)}$-reasons.

$\left(\Rightarrow \mathrm{Tor}_1^{\mathcal{O}_{U,x}}\left(\Omega^1_{U,x}, \kappa(x)\right) = 0\right.$

$\left.\Rightarrow \text{Any surjection } \mathcal{O}_{U,x}^d \twoheadrightarrow \Omega^1_{U,x} \text{ iso}\right).$

$\Rightarrow \Omega^1_{U,x}$ free rank d_x $\Rightarrow$ @ loc split. $\square$

3) $\Rightarrow$ 4) @ locally split $\Rightarrow$ Ω^1_U, I/I^2 loc free

w/ $\operatorname{rk} \Omega^1_{U,x} + \operatorname{rk}(I/I^2)_x = n \quad \forall x \in U$.

Krull principal ideal Thm $\Rightarrow$ $\operatorname{rk}(I/I^2)_x \geq n - d_x$

For $x \in U$ closed, have $\operatorname{rk} \Omega^1_{U,x} = \dim \mathfrak{m}_x/\mathfrak{m}_x^2 \geq d$.

$\Rightarrow$ Equality in both cases.

Geb:

$$\begin{tikzcd}
0 \arrow[r] & I/I^2 \arrow[r, "\alpha"] & i^*\Omega^1_{\mathbb{A}^n} \arrow[r] & \Omega^1_U \arrow[r] & 0 \\
& \mathcal{O}_U^{\oplus r} \arrow[u, two heads, "(g_j)"] \arrow[r, "\left(\frac{\partial g_j}{\partial T_i}\right)"'] & \bigoplus_{i=1}^n \mathcal{O}_U \cdot dT_i \arrow[u, "\wr"'] & &
\end{tikzcd}$$

$\Rightarrow$ $\square$

4) $\Rightarrow$ 1) Assumption implies $\dim_{\kappa(x)}\left(\kappa(x) \otimes \Omega^1_{U,x}\right) = d_x \quad \forall x$.

& $\dim_{\kappa(x)}\left(\kappa(x) \otimes \ker \alpha\right) = n - d_x$

$\rightarrow$ Local freeness of $\Omega^1_{U,x}$ as in 2) $\Rightarrow$ 3). $\square$

<u>Cor</u> If $X \to \operatorname{Spec} k$ smooth, then X reduced & locally integral. In ptic, irred comp = connected comp.

<u>Pf.</u> Properties of regular rings. $\square$

§3 Analytification revisited

Last time: $\left\{ \begin{matrix} \text{separated smooth} \\ \text{1-dim } X/\mathbb{C} \end{matrix} \right\} \longrightarrow \{\text{Riemann Surf.}\}$

$$(X, \mathcal{O}_X) \longmapsto (X^{an}, \mathcal{O}_{X^{an}}).$$

.) $X^{an} := X(\mathbb{C})$ + top from $U(\mathbb{C}) \subseteq \mathbb{C}^n$

$$X \supseteq U \text{ open}, \quad U \hookrightarrow \mathbb{A}^n.$$

.) $\mathcal{O}_{X^{an}}(V) = \{ f : V \to \mathbb{C},\ \exists\ V = \cup W_i$ s.th.

$f|_{W_i} = \varphi_i \circ g_i$, $g_i \in \mathcal{O}_X(U_i)$, $W_i \subseteq U_i^{an}$

(*) φ_i holomorphic. $\}$

Prop The so-defined loc ringed space X^{an} is indeed a R.S., i.e. locally $\cong (V, \mathcal{O}_V)$ for $V \subseteq \mathbb{C}$ open.

Proof Enough to show for U^{an}, $U \subseteq X$ affine open.

Choose $U \hookrightarrow \mathbb{A}^n_{\mathbb{C}}$. By 3) & 4) of prev. Prop, after Zariski-localizing, may assume $U = V(g_2, \dots, g_n)$ w/ $\left(\frac{\partial g_j}{\partial T_i} \right)_{\substack{j=2,\dots,n \\ i=1,\dots,n}}$ of maximal rank $n-1$ everywhere.

Localizing further, wlog $\Omega^1_U \simeq \mathcal{O}_U$.

$U_i := D(dT_i|_U)$ principal opens covering U.

<u>Implicit Fct Thm</u> Compositions $U_i \hookrightarrow \mathbb{C}^n \xrightarrow[=T_i]{pr_i} \mathbb{C}$

are local isomorphisms.

At same time, T_i <u>algebraic</u>, i.e. may be used as g in $(*)$. $\square$

<u>Rmk</u> X^{an} comes w/ natural map (of loc ringed sp)

$$(X^{an}, \mathcal{O}_{X^{an}}) \longrightarrow (X, \mathcal{O}_X)$$

Universal property:

$$\begin{tikzcd} (X^{an}, \mathcal{O}_{X^{an}}) \arrow[r] & (X, \mathcal{O}_X) \\ (Y, \mathcal{O}_Y) \arrow[u, dashed, "\exists!"] \arrow[ur, "\forall \text{ map of loc. ringed sp.}"'] & \end{tikzcd}$$

$\forall$ R.S. $(Y, \mathcal{O}_Y)$

Determines $X^{an} \longrightarrow X$ unique up to unique iso.

Try this yourself!

(<u>Hint</u>: $\mathrm{Mor}(X, \mathrm{Spec}\, A) = \mathrm{Rg.homo}(A, \mathcal{O}_X(X))$

for <u>all</u> loc. ringed sp X, not just schemes.)